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Complexes, Group Cohomology, and an Induction Theorem for the Green Ring

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FOR SANDY GREEN ON HIS 60TH BIRTHDAY

We prove a strengthened version of Theorem A of [4]. The proof is quicker than the previous proof and it also gives very good insight into what is going on. In some ways this proof is less elementary. As in [4] we fix a prime p, let $A(G)$ denote the Green ring of $\mathbb{Z}_pG$-modules, where $\mathbb{Z}_p$ denotes the p-adic integers, and let $\mathscr{C} = \{H \leq G \mid H/O_p(H) \text{ is cyclic}\}$ be the set of subgroups of G which are cyclic mod p.

THEOREM. *Let G be a finite group acting simplicially on a finite simplicial complex Δ. Suppose that for all $\sigma \in \Delta$ the isotropy group G_σ fixes σ pointwise. Then the following conditions are equivalent.*

(i) *For all $H \in \mathscr{C}$ with $p \mid |H|$, $\chi(\Delta^H) = 1$.*

(ii) *In $A(G)$, $\mathbb{Z}_p \equiv \sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma} \mathbb{Z}_p \uparrow_{G_\sigma}^{G}$* (mod projectives).

(iii) *For all finitely generated $\mathbb{Z}G$-modules M and integers n,*

$$\hat{H}^n(G, M)_p = \sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma} \hat{H}^n(G_\sigma, M)_p.$$

In fact condition (iii) is equivalent to the same condition for just one value of n. This equation holds in the Grotendieck group $K_0(Ab, 0)$ of finite abelian groups with respect to direct sum decompositions, and the suffix p means the p-torsion subgroup. In [4] we obtained the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and these are really the useful ones for the purposes of cohomology, but it is interesting to see that they may be reversed, so that the fixed-point condition (i) is exactly what we need for our cohomology calculations. We refer the reader to [4] for the applications to standard simplicial complexes associated to G. The simplicial complexes of all p-subgroups of G, all elementary abelian p-subgroups of G, and the Tits building in the case of a Chevalley group all satisfy condition (i), and hence may be used to compute cohomology.

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The main work of the theorem is in the equivalence (i) $\Leftrightarrow$ (ii), and this depends fundamentally on Conlon's induction theorem. This was also the main ingredient in [4] but the approach here is faster and more satisfying conceptually. One can now see that it is no surprise to obtain a cohomology formula looking like an Euler characteristic when one has already started with a hypothesis about Euler characteristics. The implication (ii) $\Rightarrow$ (iii) is not particularly difficult and proceeds by applying the functor Ext^n to both sides of (ii), as in [4]. The reverse implication (iii) $\Rightarrow$ (ii) is harder and uses a theorem of Benson [1] concerning the radical of dim Ext^n (,), viewed as a bilinear form on the Green ring.

The induction theorem of the title is statement (ii) of the theorem. While the statement is only a congruence, the coefficients of the induced modules are at least given in quite a striking way. It is perhaps worth pointing out that a theorem in my previous paper [4], Theorem D′, is also an induction theorem for $A(G)$, although this is not discussed there. It is really the explicit form of Conlon's induction theorem. In Conlon's original theorem an assertion is only made about the existence of an expression for $\mathbb{Z}_p$ as a rational linear combination of modules induced from subgroups in $\mathscr{C}$. In Theorem D′ of [4] these rational coefficients are made explicit in terms of Möbius functions. This is entirely analogous to the way the coefficients are given in Brauer's proof of Artin's induction theorem for ordinary characters, which is again in terms of Möbius functions.

Part of the approach taken in this paper arose after a conversation with K. S. Brown, and I would like to thank him and Cornell University for hospitality.

1. The Equivalence of (i) and (ii)

We take the Green ring $A(G) = K_0 \pmod{\mathbb{Z}_pG, 0} \otimes_{\mathbb{Z}} \mathbb{C}$ to be the complex vector space with the isomorphism classes of finitely generated indecomposable $\mathbb{Z}_pG$-modules as a basis. We work with a complex vector space here purely for reasons of technical convenience, because evidently condition (ii) holds in $K_0 \pmod{\mathbb{Z}_pG, 0}$. To apply Conlon's induction theorem we first set up some notation to do with characters and species of the Green ring. We will only make the definitions for permutation modules, so suppose that Ω is a G-set and let $M = \mathbb{Z}_p\Omega$ be the corresponding permutation module with ordinary character ϕ_M. If g is any element of G then $\phi_M(g)$ is the trace of g acting on M, and this is the number of fixed points of g on Ω, i. e., $|\Omega^{\langle g \rangle}|$. The point about this is that the value is independent of the choice of generator of $\langle g \rangle$, and with this in mind we define a "character" taking values on subgroups of G by $\phi_M(H) = |\Omega^H|$, the number of fixed points of H on Ω. We might as well say straight away that this new character is not well defined on all subgroups of G, because it is possible to have inequivalent G-sets with isomorphic permutation modules, so that the definition depends really on Ω rather than M. However, when $H \in \mathscr{C}$ it turns out that this function is well defined. This will follow from the first lemma, but to establish this somewhat technical point we really need to use certain further concepts associated to $A(G)$. We will lift the notation for them directly from [1], and if the reader has not met these things before it would be quite all right for him to skip immediately to the statement of Theorem 1.2 and then continue with the text at the end of its proof.

We recall from [1, p. 65] the definition of the species of the trivial source subring $A(G, \text{Triv})$ of $A(G)$. These are denoted $s_{H,b}$ where $H \in \mathscr{C}$ and b is a generator of $H/O_p(H)$. If M is a trivial source module (a summand of a permutation module) we may write

$$M\downarrow_H = M_1 \oplus M_2,$$

where the summands of M_1 have vertex $O_p(H)$ and the summands of M_2 have smaller vertex. Then $O_p(H)$ acts trivially on M_1, and if θ is the Brauer character of M_1 as an $H/O_p(H)$-module then $s_{H,b}(M) = \theta(b)$.

1.1. LEMMA. *Let $H \in \mathscr{C}$ be cyclic* mod p *and suppose that $H/O_p(H) = \langle b \rangle$. Then $\phi_M(H) = s_{H,b}(M)$, where $s_{H,b}$ is the species of the trivial source subring of $A(G)$.*

Proof. If $M = \mathbb{Z}_p\Omega$ then M_1 is the permutation module on the H-orbits of Ω which $O_p(H)$ fixes. Thus the value of the species on b is the trace of b on this permutation module, which is the number of fixed points of $\langle O_p(H), b \rangle = H$ on Ω.

The use of this extended notion of a character for permutation modules comes from a form of Conlon's induction theorem, which we will essentially need to quote.

1.2. THEOREM. *Let M and N be permutation modules defined over $\mathbb{Z}_p$. Then*

(i) $\phi_M(H) = \phi_N(H)$ *for all $H \in \mathscr{C}$ if and only if $M \cong N$.*

(ii) $\phi_M(H) = \phi_N(H)$ *for all $H \in \mathscr{C}$ with $p \mid |H|$ if and only if $M \cong N$* (mod projectives)

Proof. Part (i) is 2.13.1 of [1] combined with Lemma 1.1. Part (ii) arises from the following more general observation coming from the proof of (i). For any class $\mathfrak{X}$ of subgroups of G which is closed under conjugation and forming subgroups let

$$A_{\mathfrak{X}} = \langle M \mid M \text{ has trivial source and is projective relative to } \mathfrak{X} \rangle$$

as a subspace of $A(G)$. Then $A_{\mathfrak{X}}$ is precisely the subspace of $A(G, \text{Triv})$ on which all of the species $\{s_{H,b} | O_p(H) \notin \mathfrak{X}\}$ vanish. For by construction these species do all vanish there, and on the other hand the argument for 2.13.1 of [1] shows that any indecomposable module outside $A_{\mathfrak{X}}$ is distinguishable from the other basis elements by these species, i. e., these species span the dual space of $A(G, \text{Triv})/A_{\mathfrak{X}}$. Hence any two trivial source modules M and N are congruent modulo $A_{\mathfrak{X}}$ if and only if $s_{H,b}(M) = s_{H,b}(N)$ for all $s_{H,b}$ with $O_p(H) \notin \mathfrak{X}$. In our case we take $\mathfrak{X}$ to consist of the identity subgroup, when $A_{\mathfrak{X}}$ is the span of the projective modules and $O_p(H) \notin \mathfrak{X}$ means $p \,||H|$, since $H \in \mathscr{C}$. If M and N are permutation modules then $s_{H,b}(M) = \phi_M(H)$ as previously observed, so we obtain (ii).

Using the fact that ϕ_M is additive in M we extend its definition to virtual modules. If $V = \sum \lambda_i M_i$ is any element of $A(G)$, where the M_i are modules, we define

$$\phi_V(H) = \sum \lambda_i \phi_{M_i}(H).$$

Evidently part (ii) of 1.2 will hold for virtual modules as well as actual ones. The particular element of $A(G)$ which we are interested in arises from the simplicial complex Δ on which G acts in the main theorem. If $C_i(\Delta)_p$ denotes the free $\mathbb{Z}_p$-module on the simplices in dimension i, we put

$$[\Delta] = \sum_{i=1}^{d} (-1)^i C_i(\Delta)_p \in A(G),$$

where $d = \dim \Delta$.

The proof of the equivalence (i) $\Leftrightarrow$ (ii) is now quite straightforward. Note that the right-hand side in (ii) is just $[\Delta]$, since to get the G-module structure of $C_i(\Delta)$ we just divide the simplices in dimension i into orbits, and each orbit contributes a permutation module $\mathbb{Z}_p \uparrow_{G_\sigma}^{G}$. Thus (ii) says $\mathbb{Z}_p \equiv [\Delta]$ (mod projectives). By 1.2 (ii) this is equivalent to $\phi_{\mathbb{Z}_p}(H) = \phi_{[\Delta]}(H)$ for all $H \in \mathscr{C}$ with $p \mid |H|$. Now $\phi_{\mathbb{Z}_p}(H) = 1$, and from the definition

$$\begin{aligned} \phi_{[\Delta]}(H) &= \sum_{i=1}^{d} (-1)^i \ \text{(no. of fixed points in dimension } i) \\ &= \chi(\Delta^H), \qquad \text{the Euler characteristic.} \end{aligned}$$

Condition (ii) is thus seen to be equivalent to the condition that the Euler characteristic is 1.

2. The Equivalence of (ii) and (iii)

The implication (ii)$\Rightarrow$(iii) follows exactly as in [4] by applying $\mathrm{Ext}^n_{\mathbb{Z}_pG}(\ ,M_p)$ to both sides of the congruence in (ii). Then the congruence becomes an equality because projective modules give zero when Ext is applied, and each of the permutation modules becomes a cohomology group in the following manner:

$$\mathrm{Ext}^n_{\mathbb{Z}_pG}(\mathbb{Z}_p\uparrow^G_{G_\sigma}, M_p)=\mathrm{Ext}^n_{\mathbb{Z}_pG_\sigma}(\mathbb{Z}_p, M_p)$$

$$=\mathrm{Ext}^n_{\mathbb{Z}G_\sigma}(\mathbb{Z}, M)_p=H^n(G_\sigma, M)_p.$$

The fact that we can take completion at p outside the Ext group follows from p. 233 of [2].

For the converse implication (iii)$\Rightarrow$(ii) we use a theorem of Benson on the radical of the bilinear form dim Ext^n. Because this theorem has to do with modular representations instead of p-adic representations, we will also need to use the fact that trivial source modules are liftable. We first state Benson's theorem. Let $k=\mathbb{Z}/p\mathbb{Z}$ and let $A(kG)$ now denote the Green ring of finitely generated kG-modules. Benson [1, p. 91] defines bilinear forms $(\ ,\ )_n$ on $A(kG)$ as follows. If U and V are indecomposable kG-modules (i. e., basis elements of $A(kG)$) we put

$$(U, V)_n=\dim_k \mathrm{Ext}^n_{kG}(U, V)$$

and we extend this bilinearly to give a bilinear form on the whole of $A(kG)$. The *radical* of the form is defined to be

$$\mathrm{Rad}\ (\ ,\ )_n=\{x\in A(kG)\,|\,(x, y)_n=0 \text{ for all } y\in A(kG)\}.$$

2.1. Theorem [1, p. 92]. *$\mathrm{Rad}(\ ,\ )_n$ is the linear span in $A(kG)$ of the projective modules and elements of the form*

$$\sum_{i=1}^{2s}(-1)^i\,\Omega^i(M),$$

where M is a periodic module of even period $2s$. Here Ω denotes the Heller operator.

We translate between $\mathbb{Z}_p$ and $\mathbb{Z}/p\mathbb{Z}$ representations by the following lemma.

2.2. LEMMA. *Condition* (ii) *in the main theorem is equivalent to*

$$k \equiv \sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma} k\uparrow_{G_\sigma}^{G} \quad (\text{mod projective } kG\text{-modules}).$$

Proof. Evidently (ii) implies the above condition by reducing mod p. Conversely suppose that the above modular condition holds. This means there is an isomorphism

$$k \oplus \sum_{\substack{\sigma \in \Delta/G \\ \dim \sigma \text{ odd}}} k\uparrow_{G_\sigma}^{G} \oplus P \to \sum_{\substack{\sigma \in \Delta/G \\ \dim \sigma \text{ even}}} k\uparrow_{G_\sigma}^{G} \oplus Q$$

for certain projective modules P and Q. By [3,II,12.4] this isomorphism lifts to a homomorphism

$$\mathbb{Z}_p \oplus \sum_{\substack{\sigma \in \Delta/G \\ \dim \sigma \text{ odd}}} \mathbb{Z}_p\uparrow_{G_\sigma}^{G} \oplus \hat{P} \to \sum_{\substack{\sigma \in \Delta/G \\ \dim \sigma \text{ even}}} \mathbb{Z}_p\uparrow_{G_\sigma}^{G} \oplus \hat{Q},$$

where $\hat{P}$ and $\hat{Q}$ are the lifts of P and Q. By Nakayama's lemma this is an isomorphism, so that condition (ii) follows.

We now present the proof of the implication (iii) $\Rightarrow$ (ii). Let x be the element

$$x = -k + \sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma} k\uparrow_{G_\sigma}^{G} \in A(kG).$$

Assuming (iii) we have that for any integer $n \geqslant 1$ and any kG-module M,

$$\begin{aligned}(x, M)_n &= -\operatorname{Ext}_{kG}^n(k, M) + \sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma} \operatorname{Ext}_{kG}^n(k\uparrow_{G_\sigma}^{G}, M) \\ &= -H^n(G, M) + \sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma} H^n(G_\sigma, M) \\ &= 0.\end{aligned}$$

Hence $x \in \operatorname{rad}(\ ,\ )_n$ and so by Benson's theorem we may write

$$x = \sum_P \lambda_P P + \sum_M \mu_M \omega_M,$$

where the P's are projective modules and where

$$\omega_M = \sum_{i=1}^{2s} (-1)^i \Omega^i(M),$$

M periodic of even period $2s$. We may assume that the M's shown in the expression for x all belong to distinct Ω orbits, and then because all other modules in the expression are trivial source modules, it follows by linear independence of the basis elements of $A(kG)$ that all the $\Omega^i(M)$ also have trivial source. It will complete the proof to show that these modules M do not really occur, that is $\mu_M = 0$ for all M. This will follow from the final lemma since M must have an even period.

2.3. Lemma. *Suppose that M and $\Omega(M)$ are both trivial source modules. Then k has characteristic 2 and M is periodic with respect to Ω of period 1.*

Proof. We may assume that M is indecomposable. If V is a vertex of M then M is a summand of $k\uparrow_V^G$, k is a source and k is a summand of $M\downarrow_V$. Regarding all modules as V-modules, it follows that $\Omega(k)$ is a summand of $\Omega(M)$, and hence $\Omega(k)$ is a trivial source V-module. Since $\Omega(k)$ and k have the same vertex [3, p. 105, Ex. 1], $\Omega(k)$ is a summand of $k\uparrow_V^V = k$ [3, II,12.5], so $\Omega(k) = k$. Because V is a p-group, $|V| = \dim k + \dim \Omega(k) = 2$, so $V = C_2$ and char $k = 2$. We show that in this case M must have period 1.

It suffices to show that the Green correspondent $f(M)$ has period 1 as a module for $H = N_G(V)$. Now since $f(M)$ has vertex $C_2 \cong V \lhd H$ it follows that V acts trivially on $f(M)$ (by an argument with Mackey decompositions) so $f(M)$ is a projective H/V-module made into an H-module. It thus has a simple top and socle which are isomorphic, and the same goes for $\Omega f(M)$. In the projective cover sequence

$$0 \to \Omega f(M) \to P_{f(M)} \to f(M) \to 0$$

the two left-hand modules have the same socle and the two right-hand modules have the same top, and these are all the same since $P_{f(M)}$ is projective. It follows that $f(M)$ and $\Omega f(M)$ are the same projective H/V-module, so M has period 1.

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